THE AVERAGE OF THE VALUES OF A FUNCTION AT RANDOM POINTS

BY

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ABSTRACT

The behavior of $(1/N)\ \Sigma_{n=1}^{N} f(S_n)$ as $N \to \infty$ is considered, where f is a bounded measurable function on $(-\infty, \infty)$ and $(S_n)_{n=1}^{\infty}$ are the partial sums of a sequence of independent and identically distributed random variables.

Let f be a bounded, Borel measurable function on the real line. Let $X = (X_1, X_2, \cdots)$ be a sequence of independent and identically distributed random variables. Denote by $S = (S_1, S_2, \cdots)$ the partial sums $S_n = X_1 + X_2 + \cdots + X_n$. This paper studies the limiting behavior of $(1/N)\ \Sigma_{n=1}^{N} f(S_n)$ as $N \to \infty$. For simplicity, we will now introduce the main results under the further assumption that the distribution of X_1 is not singular to Lebesgue measure. The singular (lattice and not lattice) case will also be discussed.

RESULT 1. *If* $0 < E(X_1) < \infty$, *then as* $N \to \infty$,

$$(1)\qquad (1/N)\sum_{n=1}^{N} f(S_n) - (1/(NE(X_1)))\int_0^{NE(X_1)} f(y)dy \to 0 \qquad a.s.$$

As a corollary, observe that

$$(2)\qquad \limsup_{N\to\infty}\ ((1/N)\sum_{n=1}^{N} f(S_n)) = \limsup_{N\to\infty}\ ((1/N)\int_0^{N} f(y)dy) \qquad a.s.$$

Even if the value of $E(X_1)$ is not directly involved, (2) might not hold when $E(X_1) = \infty$, as shown by part (b) of the next result.

As for the case $E(X_1) = 0$, nothing like Result 1 can be obtained in general. Consider the function $f(x) = 0$ or 1 depending on whether x is negative or non-

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negative. By the arc sine law, $(1/N)\ \Sigma_{n=1} f(S_n)$ has in some cases a nondegenerate asymptotic distribution.

RESULT 2. *Assume f to be nonnegative.*

a) *If* $\limsup_{N\to\infty}((1/N)\sup_M \int_M^{M+N} f(y)dy) = 0$, *then*

$$\limsup_{N\to\infty}((1/N)\sum_{n=1}^{N} f(S_n)) = 0 \qquad a.s.$$

b) *If* $\limsup_{N\to\infty}((1/N)\sup_{M\geq 0}\int_M^{M+N} f(y)dy) > 0$, *then there exist distributions for X_1 for which $P(X_1 > 0) = 1$ and*

$$\limsup_{N\to\infty}((1/N)\sum_{n=1}^{N} f(S_n)) > 0 \qquad a.s.$$

* * *

X is said to be *distributed on (the lattice)* $L_d = \{0, \pm d, \pm 2d, \cdots\}$ $(d > 0)$, if $P(X_1 \in L_d) = 1$ and d is the largest number with this property. If no such d exists, X is *nonlattice*. F is a distribution on L_d if, for X distributed F, X is distributed on L_d.

(3) Denote by $C(K)$ the cardinality of the set K and by $L(K)$ the Lebesgue measure of the set K. I will denote the set of positive integers.

LEMMA 1. *Assume X_1 is distributed on L_1 and $0 < E(X_1) < \infty$. Then*

$$(4)\quad \limsup_{N\to\infty}\left[\sup_K \left|(1/N)\sum_{n=1}^{N} P(S_n \in K) - (1/NE(X_1))\,C(K\cap(0, NE(X_1)))\right|\right] = 0.$$

The supremum is taken over all subsets K of the set I of positive integers.

PROOF. Denote $\theta = E(X_1)$, $\gamma = 1 + \Sigma_{n=1}^{\infty} P(S_n = 0)$. $\gamma < \infty$ because $\theta > 0$. Fix $0 < \varepsilon < \theta$.

$$(5)\quad \begin{aligned}(1/N)\sum_{n=1}^{N} P(S_n\in K) &= (1/N)\sum_{\alpha\in K} E(C\{n\,|\,1\leqq n\leqq N, S_n=\alpha\})\\ &= (1/N)\sum_{\alpha\in K\cap(0,(\theta-\varepsilon)N)} E(C\{n\,|\,n\geqq 1,\ S_n=\alpha\})\\ &\quad - (1/N)\sum_{\alpha\in K\cap(0,(\theta-\varepsilon)N)} E(C\{n\,|\,n>N,\ S_n=\alpha\})\\ &\quad + (1/N)\sum_{\alpha\in K\cap[(\theta-\varepsilon)N,(\theta+\varepsilon)N]} E(C\{n\,|\,1\leqq n\leqq N,\ S_n=\alpha\})\\ &\quad + (1/N)\sum_{\alpha\in K\cap((\theta+\varepsilon)N,\infty)} E(C\{n\,|\,1\leqq n\leqq N,\ S_n=\alpha\}).\end{aligned}$$

We will now study each summand in the right hand side of (5) separately.

By the renewal theorem, (see [2, p. 347]),

$$E(C\{n \mid n \geqq 1,\ S_n = \alpha\}) \to 1/\theta \text{ as } \alpha \to \infty. \tag{6}$$

Hence

$$\sup_K \Big| (1/N) \sum_{\alpha \in K \cap (0,(\theta-\varepsilon)N)} E(C\{n \mid n \geqq 1, S_n = \alpha\}) - (1/N\theta)\, C(K \cap (0,(\theta-\varepsilon)N)) \Big| \to 0 \tag{7}$$

as $N \to \infty$.

Secondly,

$$\begin{aligned} &(1/N) \sum_{\alpha \in K \cap (0,(\theta-\varepsilon)N)} E(C\{n \mid n > N, S_n = \alpha\}) \\ &\leqq (\gamma/N) \sum_{\alpha < (\theta-\varepsilon)N} P(\alpha \in \{S_{N+1}, S_{N+2}, \cdots\}) \\ &\leqq (\gamma/N) \sum_{\alpha < (\theta-\varepsilon)N} P(\min(S_{N+1}, S_{N+2}, \cdots) \leqq \alpha) \\ &\leqq \gamma(\theta-\varepsilon)\, P((1/N)\min(S_{N+1}, S_{N+2}, \cdots) < \theta - \varepsilon) \to 0 \text{ as } N \to \infty, \end{aligned} \tag{8}$$

by the strong law of large numbers.

Thirdly,

$$(1/N) \sum_{\alpha \in K \cap [(\theta-\varepsilon)N, (\theta+\varepsilon)N]} E(C\{n \mid 1 \leqq n \leqq N, S_n = \alpha\}) \leqq \gamma(2\varepsilon + (1/N)), \tag{9}$$

and in the fourth term,

$$\begin{aligned} &\sum_{\alpha \in K \cap ((\theta+\varepsilon)N, \infty)} E(C\{n \mid 1 \leqq n \leqq N,\ S_n = \alpha\}) \\ &\leqq (1/N) \sum_{\alpha > (\theta+\varepsilon)N} E(C\{n \mid 1 \leqq n \leqq N,\ S_n = \alpha\}) \\ &= (1/N)\, E(C\{n \mid 1 \leqq n \leqq N,\ S_n > (\theta+\varepsilon)N\}) \\ &\leqq P((1/N)\max(S_1, S_2, \cdots, S_N) > \theta + \varepsilon) \to 0 \quad \text{as} \quad N \to \infty \end{aligned} \tag{10}$$

by the strong law of large numbers.

Combine (5), (7), (8), (9) and (10) to obtain

$$\limsup_{N \to \infty} \left[\sup_K \Big| (1/N) \sum_{n=1}^{N} P(S_n \in K) - (1/N\theta)\, C(K \cap (0, N\theta]) \Big| \right] \leqq \varepsilon(2\gamma + 1/\theta). \tag{11}$$

Since ε is arbitrary, (4) follows.

LEMMA 2. *Assume* X_1 *is nonlattice, and* $0 < E(X_1) < \infty$.

Let U be independent of X, uniformly distributed is some interval $[a,b]$. Then

$$\limsup_{N\to\infty}\Big[\sup_K \Big|(1/N)\sum_{n=1}^{N} P(U+S_n\in K) - (1/NE(X_1))L(K\cap(0,NE(X_1)])\Big|\Big]=0. \tag{12}$$

The supremum is taken over all measurable subsets K of $(0,\infty)$.

PROOF.

$$P(U+S_n\in K) = \frac{1}{b-a}\int_K P(\alpha-b<S_n\leqq\alpha-a)d\alpha.$$

Hence

$$(1/N)\sum_{n=1}^{N} P(U+S_n\in K) = (1/N(b-a))\int_K E(C\{n\mid 1\leqq n\leqq N,\ \alpha-b<S_n\leqq\alpha-a\})\,d\alpha.$$

From this point on, the proof follows the same lines as that of Lemma 1.

THEOREM 1. *Assume* $0<E(X_1)<\infty$.

a) *If X_1 is distributed on L_d, then*

$$(1/N)\sum_{n=1}^{N} f(S_n) - (d/NE(X_1))\sum_{n=1}^{[NE(X_1)/d]} f(nd)\to 0 \qquad a.s.\ as\ N\to\infty. \tag{13}$$

b) *If X_1 is nonlattice, then for all x outside a Lebesgue null set,*

$$(1/N)\sum_{n=1}^{N} f(x+S_n) - (1/NE(X_1))\int_0^{NE(X_1)} f(y)dy\to 0 \qquad a.s.\ as\ N\to\infty. \tag{14}$$

c) *If the distribution of X_1 is not singular to Lebesgue measure, then*

$$(1/N)\sum_{n=1}^{N} f(S_n) - (1/NE(X_1))\int_0^{NE(X_1)} f(y)dy\to 0 \qquad a.s.\ as\ N\to\infty. \tag{15}$$

PROOF.

Part (a). Without loss of generality, assume that $d=1$, and that f only assumes the values 0 and 1. The statement for more general f's can then be obtained by approximating f by simple functions.

Denote $F=\{n\mid n\in I,\ f(n)=1\}$. Denote $\theta=E(X_1)$. Fix $0<\varepsilon<\theta$. Let the positive integer M satisfy (16) and (17).

$$\sup_{K\subset I}\Big|(1/M)\sum_{n=1}^{M} P(S_n\in K) - (1/M\theta)\,C(K\cap(0,M\theta))\Big| < \varepsilon/2. \tag{16}$$

$$E\left|(S_M/M)-\theta\right|\leqq \theta\varepsilon/2. \tag{17}$$

Such an M exists, because S_n/n as a reversed martingale converges to θ in the L_1 norm (see [1, Th. 5.24 and Problem 5.7]). Denote $S_0=0$. Let $N=mM$.

$$\begin{aligned}(1/N)&\sum_{n=1}^{N} f(S_n)-(1/N\theta)\sum_{n=1}^{[N\theta]} f(n)\\
&= (1/m)\sum_{n=1}^{m}(1/M)[C\{i\,|\,(n-1)M<i\leqq nM,\ S_i\in F\}\\
&\qquad -E(C\{i\,|\,(n-1)M<i\leqq nM,\ S_i\in F\}\,|\,S_0,S_1,\cdots,S_{(n-1)M})]\\
&+(1/m)\sum_{n=1}^{m}(1/M)[E(C\{i\,|\,(n-1)M<i\leqq nM,\ S_i\in F\}\,|\,S_0,\cdots,S_{(n-1)M})\\
&\qquad -(1/\theta)C\{i\,|\,0<i-S_{(n-1)M}\leqq M\theta,\ i\in F\}]\\
&+(1/m)\sum_{n=1}^{m}(1/M\theta)[C\{i\,|\,0<i-S_{(n-1)M}\leqq M\theta,\ i\in F\}\\
&\qquad -C\{i\,|\,S_{(n-1)M}<i\leqq S_{nM},\ i\in F\}]\\
&+(1/N\theta)[C\{i\,|\,0<i\leqq S_N,\ i\in F\}-C\{i\,|\,0<i\leqq N\theta,\ i\in F\}].\end{aligned} \tag{18}$$

In the last two summands of the right hand side of (18), if an inequality is vacuous, reverse it and change the sign of the corresponding $C\{.\}$. The first summand converges to zero a.s. as $N\to\infty$, by Levy's strong law of large numbers for martingales with bounded increments, (see [3 p. 250] or [4 p. 146]). The second summand is bounded in absolute value by $\varepsilon/2$, by (16).
The third summand is bounded in absolute value by

$$(1/m\theta)\ \textstyle\sum_{n=1}^{m}\left|(S_{nM}-S_{(n-1)M})/M-\theta\right|.$$

As $N\to\infty$, this bound converges a.s. to $(1/\theta)E\left|S_M/M-\theta\right|\leqq\varepsilon/2$, by (17) and the strong law.

The last summand is bounded in absolute value by $(1/\theta)\left|S_N/N-\theta\right|$, which converges a.s. to zero, by the strong law.
Hence

$$\limsup_{m\to\infty}\left|(1/N)\sum_{n=1}^{N} f(S_n)-(1/N\theta)\sum_{n=1}^{[N\theta]} f(n)\right|\leqq\varepsilon \quad \text{a.s.} \tag{19}$$

Since the left hand side of (19) does not actually depend on M and ε is arbitrary, (13) follows.

Part (b). The proof of Part (a) can be easily adapted to show that if X_1 is nonlattice and U is independent of X, uniformly distributed on some interval,

$$(1/N)\sum_{n=1}^{N} f(U+S_n)-(1/NE(X_1))\int_0^{NE(X_1)} f(y)dy\to 0 \quad \text{a.s. as } N\to\infty. \tag{20}$$

But then, by Fubini, we obtain the desired result.

Part (c). From Part (b), if X_1 is nonlattice and U is independent of X and has an absolutely continuous distribution, (20) holds.

Now suppose the distribution of X_1 is not singular. Then the distribution of X_1 is a mixture of an absolutely continuous distribution F with mixing probability $p > 0$ and a singular distribution G with probability $1 - p$. Denote by H the distribution on $\{0,1\}$ with probabilities $1 - p$ and p respectively.

Let $Y_1, Y_2, \cdots; Z_1, Z_2, \cdots; P_1, P_2, \cdots$ be independent; the Y's distributed F, the Z's G and the P's H. Then $X_1, X_2, \cdots$ can be produced as follows. If $P_n = 1$, $X_n = Y_n$. Otherwise $X_n = Z_n$. Let T be the least n with $P_n = 1$, and $U = X_1 + X_2 + \cdots + X_T$. Denote, for $n \geqq 1$, $X'_n = X_{n+T}$. Then $X' = (X'_1, X'_2, \cdots)$ and U are independent, and U has an absolutely continuous distribution. Hence, because of Part (b), (20) holds, where the S_n's are the partial sums of X'. But the partial sums of X', when added to U, turn into the partial sums of X, thus giving (15), because $P(T < \infty) = 1$.

REMARK. Perhaps Part (b) will look clearer after considering the following example. Suppose $P(X_1 = 1) = P(X_1 = \pi) = \frac{1}{2}$. Then X_1 is nonlattice. S never leaves a countable set A, the additive semigroup generated by 1 and π. Let $f(x) = 1$ for $x \in A$, $f(x) = 0$ otherwise. Then $f(S_n) \equiv 1$ a.s. while $f = 0$ a.e., so (15) is not satisfied.

LEMMA 3. *Let F be a nondegenerate distribution on L_1 that assigns to zero positive probability. Let M be a positive integer bigger than 1. Then, on some probability space, it is possible to define M processes $S^{(1)}, S^{(2)}, \cdots, S^{(M)}$ and a positive-integer-valued random variable T such that for each $1 \leqq i \leqq M$, $S^{(i)} = (S_1^{(i)}, S_2^{(i)}, \cdots)$ are the partial sums of independent and identically F-distributed random variables; and*

(21) *whenever* $n \geqq T$, $S_n^{(i+1)} = S_n^{(i)} + 1$ *for every* $1 \leqq i \leqq M - 1$.

PROOF. We will prove the statement for $M = 2$. The general case requires nothing more than a simple inductive step. For every positive integer J, denote by F_J the distribution of $(\min(X,J))^+ - (\min(-X,J))^+$ when X is distributed F. Since the greatest common divisor of a set of integers is the greatest common divisor of the elements of some finite subset, there exists a positive integer J such that F_J is a nondegenerate distribution on L_1 (proper). Fix such a J.

Let $X = (X_1, X_2, X_3, \cdots)$ be i.i.d. with common distribution F. Let

$Y = (Y_1, Y_2, Y_3, \cdots)$ be i.i.d. with common distribution F_J. Let X and Y be independent. Define, for $n \geqq 1$, $X_n^1 = X_n$ if $|X_n| > J$, $X_n^1 = Y_n$ otherwise. Define, for $n \geqq 1$, $Z_n = X_n - X_n^1$.

Use $P(x=0)>0$ to obtain that $Z_1, Z_2, Z_3, \cdots$ are i.i.d. with a common bounded, symmetric, nondegenerate distribution G on L_1 proper. Hence their partial sums form a recurrent random walk on L_1. Hence, the least positive integer T for which $Z_1 + Z_2 + \cdots + Z_T = 1$ is almost surely defined. Define, for $n \geqq 1$, $X_n'' = X_n^1$ if $n \leqq T$, $X_n'' = X_n$ otherwise. Then $X_1'', X_2'', \cdots$ are i.i.d. with common distribution F. Define, for $n \geqq 1$, $S_n^{(1)} = X_1'' + X_2'' + \cdots + X_n''$ and $S_n^{(2)} = X_1 + X_2 + \cdots + X_n$. Then $S^{(1)} = (S_1^{(1)}, S_2^{(1)}, \cdots)$, $S^{(2)} = (S_1^{(2)}, S_2^{(2)}, \cdots)$ and T have the desired properties.

REMARK. Lemma 3 is a slight variation of a construction used in [5].

LEMMA 4. *Let K be a set of integers such that*

$$\limsup_{N\to\infty} \left(\sup_{M\in L_1} \left(C(K \cap (M, M+N])\right)/N\right) = 0 \tag{22}$$

Let F be a distribution on L_1. Denote by $S_1, S_2, S_3, \cdots$ the partial sums of i.i.d. F-distributed random variables.

Then

$$\limsup_{N\to\infty} \left(\sup_{M\in L_1} (1/N) \textstyle\sum_{n=1}^{N} P(M + S_n \in K)\right) = 0 \,. \tag{23}$$

PROOF. Fix $\varepsilon>0$. Let $N_0 \in I$ be such that for every interval $J \subset L_1$ of length at least N_0, $C(J \cap K) < (\varepsilon/4)C(J)$. Without loss of generality, we may assume that F is a nondegenerate distribution on L_1 proper. We will further assume that F assigns to zero positive probability. Otherwise, mix F with a point mass at zero, thus retarding the random walk at every point by a geometric time. It is easy to see that the statement holds for a walk iff it holds for a retardation of the walk. Let $S^{(1)}, S^{(2)}, \cdots, S^{(N_0)}$, T satisfy the result of Lemma 3 for the distribution F. Let the positive integer A satisfy $P(T > A) \leqq \varepsilon/2$. Let $L \geqq 4A/\varepsilon$ be arbitrary, $L \in I$. Let $M \in L_1$ be arbitrary. Then on the set $\{T \leqq A\}$, the number of pairs (i,j) with $1 \leqq i \leqq L$ and $1 \leqq j \leqq N_0$ for which $M + S_i^{(j)} \in K$ is at most $AN_0 + (\varepsilon/4)N_0 L \leqq (\varepsilon/2)N_0 L$. On the set $\{T > A\}$, the number of those pairs is of course at most $N_0 L$, so the expected number of those pairs is at most

$$(\varepsilon/2)N_0 L P(T \leqq A) + N_0 L P(T > A) \leqq \varepsilon N_0 L.$$

Since $S^{(1)}, S^{(2)}, \cdots, S^{(N_0)}$ are identically distributed, we obtain finally that for every $\varepsilon > 0$ there exists an $L_0 \in I$ such that if $L \geqq L_0$, $(1/L) \sum_{n=1}^{L} P(M + S_n^{(1)} \in K) < \varepsilon$ for every $M \in L_1$.

THEOREM 2. *Assume f to be nonnegative.*

(24) a) *If* $\limsup_{N\to\infty}((1/N)\sup_{M\in L_1} \sum_{n=M+1}^{M+N} f(nd)) = 0$ *and* X_1 *is distributed on* L_d $(d > 0)$, *then, with probability one,* $\limsup_{N\to\infty}((1/N) \sum_{n=1}^{N} f(S_n)) = 0$.

b) *If* $\limsup_{N\to\infty}((1/N)\sup_M \int_M^{M+N} f(y)dy) = 0$ *and* X_1 *is nonlattice, then, for almost all (Lebegue) x, with probability one,* $\limsup_{N\to\infty}((1/N)\sum_{n=1}^{N} f(x+S_n)) = 0$.

c) *Under the conditions of* (b), *if the distribution of* X_1 *is not singular to Lebesgue measure, the statement holds for* $x = 0$.

PROOF. We will only prove part (a). The ideas involved in the following proof and those used in the proof of parts (b) and (c) of Theorem 1 can be easily adapted to prove parts (b) and (c) of the present theorem. The only major novelty to be introduced is a modification of Lemma 3 for the nonlattice case. Fix a small $\varepsilon > 0$ and replace $S_n^{(i+1)} = S_n^{(i)} + 1$ in (21) by $S_n^{(i+1)} - S_a^{(i)} \in (0, \varepsilon)$. Without loss of generality, assume $d = 1$.

We will prove the statement for functions f assuming the values 0 and 1. Suppose this has already been done. Fix an arbitrary $\varepsilon > 0$. Denote by f^1 the indicator function of the set $\{n \in L_1 \mid f(n) > \varepsilon\}$. Then f^1 satisfies the assumptions on f, and assumes the values 0 and 1 only.

Hence,

$$\limsup_{N\to\infty} ((1/N) \sum_{n=1}^{N} f(S_n)) \leqq \varepsilon + (\sup_x f(x)) \limsup_{N\to\infty} ((1/N) \sum_{n=1}^{N} f^1(S_n)) = \varepsilon.$$

And that would end the proof.

For a function f obtaining the values 0 and 1 only and satisfying (24) for $d = 1$, denote $K = \{n \in L_1 \mid f(n) = 1\}$. Fix $\varepsilon > 0$. Using Lemma 4, fix N such that

$$\sup_{M\in L_1} (1/N) \sum_{n=1}^{N} P(M + S_n \in K) < \varepsilon. \tag{25}$$

Denote, for $n \geqq 1$, $Y_n = (1/N)C\{m \mid (n-1)N < m \leqq nN,\ S_m \in K\}$, and $Y_0 \equiv 0$. Denote, for $n \geqq 1$, $Z_n = Y_n - E(Y_n \mid Y_0, Y_1, \cdots, Y_{n-1})$. $\{Z_n\}$ are the increments of a martingale with mean zero and uniformly bounded increments.

(26) Hence $(1/H) \sum_{n=1}^{H} Z_n \to 0$ a.s. as $H \to \infty$. (See [3, Section 69, p. 250] or [4 p. 146]).

By (25), $0 \leqq E(Y_n \mid Y_0, Y_1, \cdots, Y_{n-1}) < \varepsilon$, hence (a.s.)

$$(27)\quad \limsup_{H\to\infty} (1/H) \sum_{n=1}^{H} f(S_n) = \limsup_{H\to\infty} (1/NH) \sum_{n=1}^{NH} f(S_n)$$

$$= \limsup_{H\to\infty} (1/H) \sum_{n=1}^{H} Y_n = \limsup_{H\to\infty} ((1/H) \sum_{n=1}^{H} Z_n$$

$$+ (1/H) \sum_{n=1}^{H} E(Y_n | Y_0, Y_1, \cdots, Y_{n-1})).$$

By (25), (26) and (27), $\limsup_{H\to\infty} (1/H) \sum_{n=1}^{H} f(S_n) \leqq \varepsilon$ for every $\varepsilon > 0$; hence $\limsup_{H\to\infty} (1/H) \sum_{n=1}^{H} f(S_n) = 0$.

The next theorem is in a certain sense a converse of Theorem 2.

THEOREM 3. *Assume f to be nonnegative.*

a) *If for some $d > 0$, $\limsup_{N\to\infty} ((1/N) \sup_{M\in L_1} \sum_{n=M+1}^{M+N} f(nd)) > 0$, then there exists a distribution F on L_d such that if $S_1, S_2, \cdots$ are the partial sums of i.i.d. F-distributed random variables, then, with probability one,*

$$\limsup_{N\to\infty} ((1/N) \sum_{n=1}^{N} f(S_n)) > 0.$$

b) *If $\limsup_{N\to\infty} (1/N) \sup_M \int_M^{M+N} f(y)dy > 0$, then there exist nonlattice distributions for which $\limsup_{N\to\infty} ((1/N) \sum_{n=1}^{N} f(x+S_n)) > 0$ almost surely for almost all x.*

PROOF. We will prove the theorem only for $d = 1$, f obtaining values 0 and 1 only and such that $\limsup_{N\to\infty} ((1/N) \sup_{M\in I} \sum_{n=M+1}^{M+N} f(n)) > 0$. (Observe the "$M \in I$" under the sup sign).

By the Hewitt-Savage zero-one law and by Fatou's lemma,

$$\limsup_{N\to\infty} (1/N) \sum_{n=1}^{N} f(S_n) \geqq \limsup_{N\to\infty} ((1/N) E\left(\sum_{n=1}^{N} f(S_n)\right) \quad \text{a.s.}$$

So it is enough to build a distribution F on I for which

$$(28)\qquad \limsup_{N\to\infty} (1/N) E\left(\sum_{n=1}^{N} f(S_n)\right) > 0.$$

The distribution F will have support $\{A_1, A_2, A_3, \cdots\}$, where $A_1, A_2, \cdots$ form an increasing sequence of positive integers. The probability assigned to A_i will be proportional to $(1/N_i)$, where $N_1, N_2, N_3, \cdots$ is another increasing sequence of positive integers.

We will now define inductively the two sequences.

(29) Before that, denote $K = \{n \in I \mid f(n) = 1\}$ and

$$\eta = \limsup_{N\to\infty} \left((1/N) \sup_{M \in I} \sum_{n=M+1}^{M+N} f(n) \right).$$

(30) Let $A_1 = 1$ and N_1 be any even positive integer such that for every $N \geqq N_1$, $\sup_{M \in I}((1/N) \sum_{n=M+1}^{M+N} f(n) < (9/8)\eta$.

Suppose that $A_1, A_2, \cdots, A_m$; $N_1, N_2, \cdots, N_m$ have been defined and both sequences are strictly increasing. Denote by F_m the distribution supported by $\{A_1, A_2, \cdots, A_m\}$ that assigns to A_i probability $1/(N_i \sum_{j=1}^{m}(1/N_j))$ $(i = 1, 2, \cdots m)$. Denote its mean by μ, and by $S_1^{(m)}, S_2^{(m)}, \cdots$ the partial sums of i.i.d., F_m-distributed random variables. Using Lemma 1, define N_{m+1} to be any even integer exceeding $2N_m$ such that whenever $N \geqq N_{m+1}/2$,

(31) $$\sup_{J \subset I} \left| (1/N) \sum_{n=1}^{N} P(S_n^{(m)} \in J) - (1/N\mu)\, c(J \cap \{n \in I \mid n \leqq N\mu\}) \right| < \eta/8.$$

Let A_{m+1} be a positive integer exceeding A_m and such that

(32) $$C(K \cap \{A_{m+1} + n \mid n \in I,\, n \leqq N_{m+1}\mu\})/(N_{m+1}\mu) > (7/8)\eta\,.$$

Such an A_{m+1} exists, by the definition of η. (Denote $Q(M,N) = (1/N)\sum_{n=M+1}^{M+N} f(n)$. We leave it to the reader to check that

$$\limsup_{M\to\infty} Q(M,N) \geqq \limsup_{N\to\infty} \limsup_{M\to\infty} Q(M,N) = \limsup_{N\to\infty} \sup_{M \in I} Q(M,N)).$$

Observe that $\sum_{i=1}^{\infty} (1/N_i) \leqq \sum_{i=1}^{\infty} (1/2)^i < \infty$, and let F be the distribution that assigns to A_i $(i = 1, 2, 3, \cdots)$ probability $1/(N_i \sum_{j=1}^{\infty}(1/N_j))$. We will now see that F satisfies (28). Fix $m \in I$. Think of it as being large. Let $X_1, X_2, \cdots, X_{N_m}$ be i.i.d., F-distributed random variables. Denote by B_m the event: {Among $X_1, X_2, \cdots, X_{N_m}$ all but one are less than A_m; the exceptional one equals A_m and its index is at most $(N_m/2)$}. The probability of B_m is

$$P(B_m) = (N_m/2)(c/N_m) \left[1 - (c/N_m) \sum_{j=m}^{\infty} (N_m/N_j) \right]^{N_m - 1},$$

where

$$c = \left[\sum_{j=1}^{\infty} (1/N_j) \right]^{-1}.$$

Since

$$\sum_{j=m}^{\infty} (N_m/N_j) < \sum_{j=0}^{\infty} 2^{-j} = 2,$$

(33) $$\liminf_{m\to\infty} P(B_m) \geqq (c/2)\exp(-2c).$$

(S_n) will denote partial sums of F-distributed variables and (T_n) partial sums of F_m-distributed variables.

$$\sum_{n=1}^{N_m} P(S_n \in K) \geqq \sum_{n=(N\ /2)+1}^{N_m} P(S_n \in K) = P(B_m) \sum_{n=(N_m/2)+1}^{N_m} P(S_n \in K / B_m)$$

(34)

$$= P(Bm) \sum_{n=(N_m/2)+1}^{N_m} P(A_m + T_n \in K) = P(B_m) \sum_{n=1}^{N_m} P(A_m + T_n \in K) - \sum_{n=1}^{(N_m/2)} P(A_m + T_n \in K)].$$

Use (31) to expand (34) further:

$$\sum_{n=1}^{N_m} P(S_n \in K) \geqq N_m P(B_m) [(1/N_m\mu) C(K \cap \{A_m + n \mid n \in I,\ n \leqq N_m\mu\}) \tag{35}$$
$$- (1/N_m\mu) C(K \cap \{A_m + n \mid n \in I,\ n \leqq \tfrac{1}{2} N_m\mu\}) - \frac{3}{16}\eta].$$

Apply (30) and (32) to (35), to obtain

$$(1/N_m) \sum_{n=1}^{N_m} P(S_n \in K) \geqq P(B_m)\left[\frac{7}{8}\eta - \frac{9}{16}\eta - \frac{3}{16}\eta\right] = P(B_m)\eta/8. \tag{36}$$

And finally, from (36),

$$\limsup_{N\to\infty} E\left((1/N)\sum_{n=1}^{N} f(S_n)\right) = \limsup_{N\to\infty} (1/N) \sum_{n=1}^{N} P(S_n \in K) \geqq c\exp(-2c)\eta/16 > 0.$$

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